Introduction to Probability and Statistics - 18.05 Spring 2008

Problem set 2 Solutions

1. Denote an outcome by T3 for "tails on the flip and a 3 on the roll" (for example). Then

$$\text{Sample space} = \{\text{ H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}\}$$
$$\text{"Moving at least 5 steps"} = \{\text{ H5, H6, T3, T4, T5, T6}\}$$
$$\Rightarrow \text{P(moving at least five steps)} = \tfrac{6}{12} = \tfrac{1}{2}$$

2. If the random variable X is the number of tries needed until he finds the right key, then X has a geometric distribution and we have:

P(first 5 tries are all wrong)

$$\begin{aligned}
&= P(X > 5) \\
&= 1 - P(X \leq 5) \\
&= 1 - (\frac{1}{5} \times \frac{4}{5}^{1-1} + \frac{1}{5} \times \frac{4}{5}^{2-1} + \frac{1}{5} \times \frac{4}{5}^{3-1} + \frac{1}{5} \times \frac{4}{5}^{4-1} + \frac{1}{5} \times \frac{4}{5}^{5-1}) \\
&= 1 - (\frac{1}{5})(1 + \frac{4}{5} + \frac{4}{5}^2 + \frac{4}{5}^3 + \frac{4}{5}^4) \\
&= 0.3277
\end{aligned}$$

Alternatively, since all tries are independent with $\frac{4}{5}$ chance of being wrong, the probability of getting them all wrong will be $(\frac{4}{5})^5 = 0.3277$

3. $F(1) = 0$ and $F(7) = 1$, since by "exceeds" we mean strictly bigger than, and rolling a die for once will get at most 6, which is not strictly bigger than 6; while rolling a die for 7 times will give a total sum at least 7 in the worst case when each roll result a 1.

For $F(2)$, instead of computing the probability of the sum exceeding-ing 6, we can compute its complement, i.e. the probability of two rolls summing up to less than or equal to 6. Let S_i denote the probability of two rolls summing to i.

$S_1 = 0$
$S_2 = \frac{1}{36}$, $(\{1,1\})$
$S_3 = \frac{2}{36}$, $(\{1,2\},\{2,1\})$
$S_4 = \frac{3}{36}$, $(\{1,3\},\{2,2\},\{3,1\})$
$\vdots$
$\Rightarrow \Sigma_{i=1}^{6} S_i = \Sigma_{i=1}^{6} \frac{i-1}{36} = \frac{15}{36}$
$\Rightarrow F(2) = 1 - \frac{15}{36} = \frac{21}{36}$

4. For each individual skier the probability of not falling in the first four turns is 0.7^4. Now, the r.v. X that counts how many skiers do not fall in the first four turns has a binomial distribution with parameters $n = 5$ and $p = 0.7^4$. So,

$$P(X = 2) = \binom{5}{2}(0.7^4)^2(1 - 0.7^4)^3 = 0.2529$$

5. This is in a same set up as the previous problem, each turn and each skier are independent.

   (a) The limit of 0.7^n as n goes to ∞ is 0, so the probability is 0.

   (b) Let X be the r.v. that is the first turn in which the skier falls. X has a geometric distribution with parameter $p = 0.7$. We use the memoryless property of geometric r.v.'s.

$$\begin{aligned} P(X = 10 + 10 | X > 10) &= P(X = 10) \\ &= (0.7)^9 \times 0.3 = 0.07\% \end{aligned}$$