18.05 Problem Set 3 Solutions

1. The integral of a probability density function is 1. Therefore,

$$\begin{aligned}
\int_0^1 ke^{-x}dx &= 1 \\
-ke^{-x}\mid_0^1 &= 1 \\
k(1-\frac{1}{e}) &= 1 \\
k &= \frac{e}{e-1} \\
k &= 1.58
\end{aligned}$$

The expectation of a continuous random variable is defined as following:

$$\begin{aligned}
E[X] &= \int xf(x)dx \\
&= \int xke^{-x}dx
\end{aligned}$$

Taking $u = x$ and $dv = e^{-x}$, we do integration by parts and get,

$$\begin{aligned}
E[X] &= 1.58\left(-xe^{-x}\mid_0^1 - \int_0^1 e^{-x}dx\right) \\
&\approx 0.418
\end{aligned}$$

2. The problem was mistated. The first interval on which the function is more than 0 should be [-3,-2] and not [-3,2] as we wrote. There is no solution the way it is written, so we will not grade this problem.

   For completeness, we will solve the problem (with the interval [-3,-2]). Recall that for a probability density function it must hold that $\int_{-\infty}^{\infty} f(x) = 1$. Now,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-3}^{-2} (cx+3)dx + \int_{2}^{3} (3-cx)dx$$
$$= 6 - \frac{10c}{2}$$

So $6 - \frac{10c}{2} = 1$, which gives $c = 1$.

3. Because X is a uniformly distributed r.v. we can apply the formulas and compute:

$$E[X] = \frac{5-0}{2} = 2.5$$
$$Var[X] = \frac{(5-0)^2}{12} = \frac{25}{3}$$

For part b, Since X and Y satisfies the relation $X + \frac{Y}{2} = 5$, i.e. $Y = 10 - 2X$,

$$E[10-2X] = 10 - 2E[X] = 5$$
$$Var[10-2X] = (-2)^2 Var[X] = \frac{100}{3}$$

For part c, Since $P(X = 3) \neq 0, P(Y = 8) \neq 0$, but the intersection of the two events has probability 0, as they cannot be happening at the same time, we get

$$P(A \cap B) \neq P(A) \times P(B) \Rightarrow \text{not independent.}$$

4. For discrete r.v. X, $E[X] = \Sigma a_i P(X = a_i)$. In this case, X is the number of games played in the series. Clearly, for $n = 1, 2$, $P(X = n) = 0$. When $n > 5$, $P(X = n) = 0$ as after the 5th game, there has to be a team that won at least three games. Therefore $X = 3, 4, 5$.
When $X = 3$, either Red Sox wins three games or Yankees wins all three games, $P[X = 3] = (0.6)^3 + (0.4)^3$
When $X = 4$, if Rex Sox wins the three games, they must have won the last game and two out of the first three games, this event happens with probability $0.6 \cdot \binom{3}{2}(0.6)^2(0.4)$. Likewise the event of the Yankees winning three out of four has probability $0.4 \cdot \binom{3}{2}(0.4)^2(0.6)$.
Together we get, $P[X = 4] = \binom{3}{2}(0.6)^3(0.4) + \binom{3}{2}(0.4)^3(0.6)$.
When $X = 5$, if Rex Sox wins three games, they must have won the last game and two out of the first four games. The probabilty for this event is $0.6 \cdot \binom{4}{2}(0.6)^2(0.4)^2$. Likewise the event of the Yankees winning three out of five is $0.4 \cdot \binom{4}{2}(0.4)^2(0.6)^2$.

Together we get, $P[X=5] = \binom{4}{2}(0.6)^3(0.4)^2 + \binom{4}{2}(0.4)^3(0.6)^2$
Therefore, $E[X] = \Sigma a_i P(X = a_i) = 4.0476$. Likewise $E[X^2] = \Sigma a_i^2 P(X = a_i) = 17.0604$, which gives, $Var[X] = E[X^2] - (E[X])^2 = 0.6773$

5. The expectation value for each roll is

$$E[X] = 1 \times p + (-2) \times (1-p) = 3p - 2$$

Thus 10 rolls will be $E[10X] = 10E[X] = 30p - 20$ We will play if the expectation value of money we can win is positive/nonnegative, $\Rightarrow 30p - 20 \geq 0, p \geq 0.667$