Introduction to Probability and Statistics - 18.05 Spring 2008

Problem set 4 - Solutions

1. For every $1 \leq i < j \leq k$ define the r.v. X_{ij} to be 1 if the i'th and j'th persons share a birthday, and 0 otherwise. Now, $E[X_{ij}] = \Pr[X_{ij} = 1] = \frac{1}{365}$. This is because given the birthday of the i'th person, the probability that the birthday of the j'th falls on any particular day (specifically on the birthday of the i'th) is equally likely and hence it is $\frac{1}{365}$. Let X be the number of pairs that have the same birthday. Then clearly $X = \sum_{1 \leq i < j \leq k} X_{ij}$. Hence by linearity of expectation,

$$E[X] = E[\sum_{1 \leq i < j \leq k} X_{ij}] = \sum_{1 \leq i < j \leq k} E[X_{ij}] = \binom{k}{2} \cdot \frac{1}{365}$$

2. Using linearity of expectation and the formula:

$$Var[X] = E[X^2] - (E[X])^2$$

we have:

$$\begin{aligned} E[Y] &= E[2X^2 + 5X + 4] \\ &= E[2X^2] + E[5X] + E[4] \\ &= 2E[X^2] + 5E[X] + E[4] \\ &= 2(Var[X] + E[X]^2) + 5 \times 2 + 4 \\ &= 2(3 + 2^2) + 10 + 4 \\ &= 28 \end{aligned}$$

3. By the definition of marginal density function,

$$f_X = \int_0^1 (4 - x - y)dy$$

$$
\begin{aligned}
&= 3\frac{1}{2} - x \\
f_Y &= \int_0^1 (4 - x - y)dx \\
&= 3\frac{1}{2} - y
\end{aligned}
$$

Remark: the original version of the pset had $4 - x - y$ instead of $2 - x - y$ (and we solved it for this case). But for the double integral to be 1 over the whole space (and hence be a joint density function) it should be $2 - x - y$. In any case, you get full credit regardless of which way you've computed it for (given of course that you computed it correctly).

4. Since Y and Z are discrete random variables, the probability mass function will be given by listing all possible values of $p(Y, Z)$.

$$
\begin{aligned}
p(Y, Z) &= \frac{1}{36} \qquad if \;\; Y = Z \\
&= \frac{2}{36} \qquad if \;\; Y > Z \\
&= 0 \qquad if \;\; Y < Z
\end{aligned}
$$

When $Y = a$, $p(Y) = p(a, 1) + p(a, 2) + p(a, 3) + p(a, 4) + p(a, 5) + p(a, 6)$ (this is a marginal mass function). Therefore,

$$
\begin{aligned}
\Pr(Y = 1) &= \frac{1}{36} \\
\Pr(Y = 2) &= \frac{3}{36} \\
\Pr(Y = 3) &= \frac{5}{36} \\
\Pr(Y = 4) &= \frac{7}{36} \\
\Pr(Y = 5) &= \frac{9}{36}
\end{aligned}
$$

$$\Pr(Y = 6) = \frac{11}{36}$$

5. Let X, Y and Z be uniformly distributed over (0,1). We computed in class that

$$\begin{aligned} f_{X+Y}(a) &= a && if\ \ 0 \le a \le 1 \\ &= 2 - a && if\ \ 1 \le a \le 2 \\ &= 0 && otherwise \end{aligned}$$

By the same approach, we get

$$f_{X+Y+Z}(a) = \int_0^1 f_{X+Y}(a - z) f_Z(z) dz = \int_0^1 f_{X+Y}(a - z) dz$$

This time we consider three cases:

$$\begin{aligned} f_{X+Y+Z}(a) &= \int_0^a (a - z) dz = \frac{a^2}{2} && if\ \ 0 \le a \le 1 \\ &= \int_{a-1}^1 (a - z) dz + \int_0^{a-1} 2 - (a - z) dz = \frac{a(2 - a)}{2} + \frac{-3 + 4a - a^2}{2} \\ &= \frac{-3 + 6a - 2a^2}{2} && if\ \ 1 \le a \le 2 \\ &= \int_{a-2}^1 2 - (a - z) dz = \frac{(3 - a)^2}{2} && if\ \ 2 \le a \le 3. \end{aligned}$$