**Introduction to Probability and Statistics - 18.05**

Problem set 7 solutions

1. Since the dataset is a realization of a random sample from a Geometric distribution with $p = 1/M$, the likelihood function is

$$\begin{aligned}
L(M) &= P(Y_1 = y_1, Y_2 = y_2, \dots, Y_n = y_n) \\
&= (1-\frac{1}{M})^{(y_1-1)}\frac{1}{M}(1-\frac{1}{M})^{(y_2-1)}\frac{1}{M}\cdots(1-\frac{1}{M})^{(y_n-1)}\frac{1}{M} \\
&= (1-\frac{1}{M})^{(-n+\Sigma_{i=1}^n y_i)}(\frac{1}{M})^n \\
l(M) &= -nln(M) + (-n+\Sigma_{i=1}^n y_i)ln(1-\frac{1}{M}) \\
\frac{d}{dM}(l(M)) &= \frac{-n}{M} + (-n+\Sigma_{i=1}^n y_i)\frac{1}{M(M-1)}
\end{aligned}$$

Which equals zero if and only if $M = \overline{y_n}$.

2. Because this is a uniform distribution, $f(x) = \frac{1}{\beta-\alpha}$. So the likelihood function is: $L(\beta) = f(x_1)f(x_2)\dots f(x_n) = \frac{1}{(\beta-\alpha)^n}$ if $\beta \geq max(x_1, x_2...x_n)$ and zero otherwise. Therefore, the MLE for $\beta$ is $max(x_1, x_2...x_n)$, because beyond the maximum, the larger the $\beta$ the smaller $L(\beta)$. I.e. it attains the maximum when $\beta = max(x_1, x_2...x_n)$. By a similar argument MLE for $\alpha$ is $min(x_1, x_2...x_n)$.

3. (a) Because we have a uniform distribution, we know that $E[Z_i] = 0$ and $Var[Z_i] = \frac{\delta^2}{3}$, thus $E[Z_i^2] = Var[Z_i] + (E[Z_i])^2 = \frac{\delta^2}{3}$. Thus by linearity of expectation
$E[X] = \frac{3}{n}(\frac{\delta^2}{3} + \dots \frac{\delta^2}{3}) = \frac{3}{n} \times n \times \frac{\delta^2}{3} = \delta^2$. So this is a unbiased estimator for $\delta^2$

   (b) Let $g(x) = -\sqrt{x}$, it can be checked by taking double derivative that this is a strictly convex function. Therefore we use Jensen's inequality

$$
\begin{aligned}
g(E[X]) &< E[g(x)] \\
-\sqrt{E[X]} &< E[-\sqrt{x}] \\
-\sqrt{\delta} &< -E[\sqrt{x}] \\
-\delta &< -E[\sqrt{x}] \\
\Rightarrow E[\sqrt{x}] < \delta &
\end{aligned}
$$

Thus it is negatively biased.

4. (a) $E[Y] = E[s\bar{Z_n} + (1-s)\bar{W_m}] = sE[\bar{Z_n}] + (1-s)E[\bar{W_m}]]$
$= s\mu + (1-s)\mu = \mu$ Therefore it is an unbiased estimator.

 (b) $MSE(Y) = Var(Y) = s^2 Var(\bar{Z_n}) + (1-s)^2 Var(\bar{W_m})$
$= s^2\frac{\sigma^2}{n} + (1-s)^2\frac{\sigma^2}{m}$
Differentiate this with respect to $s$, and set it to zero. We get $\frac{2s}{n} - \frac{2(1-s)}{m} = 0$
$\Rightarrow s = \frac{n}{n+m}$

5. (a) To get an unbiased estimate, let $E[T_c] = c(E[X_1] + \ldots E[X_n]) = \frac{1}{\lambda}$
$\Rightarrow c \times \frac{n}{\lambda} = \frac{1}{\lambda}$
$\Rightarrow c = \frac{1}{n}$

 (b)

$$
\begin{aligned}
MSE &= Var(T_c) + (E(T_c) + \theta)^2 \\
&= c^2 n \times \frac{1}{\lambda^2} + (c\frac{n}{\lambda} + \frac{1}{\lambda})^2
\end{aligned}
$$

Differentiate with respect to $c$ and set it to zero, we get $c = \frac{1}{1+n}$