Introduction to Probability and Statistics - 18.05 Spring 2008

Problem set 5 Solutions

1. (a) Let $X = X_1 + \cdots + X_{100}$. For every $1 \leq i \leq 100$, $E[X_i] = \frac{0.5+(-0.5)}{2} = 0$, and $Var[X_i] = \frac{(0.5-(-0.5))^2}{12} = \frac{1}{12}$. This implies that $E[X] = 0$ and $Var[X] = \frac{100}{12}$. Applying Chebyshev inequality we get,

$$Pr[|X| > 10] = Pr[|X - E[X]| > 10] \leq \frac{100/12}{10^2} = \frac{1}{12}$$

   (b) The average will approach the expectation value of $|X_i|$ as n goes to infinity. Since $|X_i|$ is uniformally distributed over [0, 0.5], its mean is 0.25, so the average will approach this number.

2. We have $\mu = E[X] = \frac{1}{\lambda} = 5$, $\sigma^2 = \frac{1}{\lambda^2} = 5^2$ and $\sigma = 5$. (This is because X is an exponential random variable). So

$$\begin{aligned} Pr[\Sigma_{i=1}^{100} X_i > 525] &= Pr\left[\frac{(\Sigma_{i=1}^{100} X_i) - 500}{5\sqrt{100}} > \frac{525 - 500}{5\sqrt{100}}\right] \\ &= 1 - \Phi(0.5) \\ &= 0.3085 \end{aligned}$$

3. There is an equivalent expression of Chernoff bounds that is more useful here

$$Pr[X > (1+\delta)\mu] \leq (\tfrac{e^\delta}{(1+\delta)^{(1+\delta)}})^\mu$$

In this problem, $\mu = 36$ and $\delta = \frac{1}{3}$. So the answer is 0.1639. To apply the Chernoff bound defined in class directly,

$$Pr[X > (1+\delta)\mu] \leq e^{-\delta^2\mu/3} = e^{-\frac{1}{9}}36/3 = 0.2636,$$

4. We consider the problem as a sequence of identical independent Bernoulli random variables with probability $p = \frac{2}{3}$.

   For Chernoff Bound, since $(1-\delta)\mu = \frac{n}{2}$ where $\mu = E[X] = np = \frac{2n}{3}$. We get $\delta = \frac{1}{4}$
   So the probability of winning less than half the games is at most $e^{-\delta^2\mu/3} = e^{-n/72}$. For $n = 30, 100, 10^6$ we get the following bounds: $0.65, 0.25, e^-13889$, respectively.

   For Chebyshev inequality, $Pr[|X - E[X]| > |\frac{n}{2} - \frac{n}{3}| = \frac{n}{6}] \leq \frac{np(1-p)}{(\frac{n}{6})^2} = \frac{8}{n}$ this gives: 0.267, 0.08, 8×10^{-6}, respectively.

   So one can see that as n grows the bound given by Chernoff is much better.