GENERATING FUNCTIONS*

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I. ENUMERATION

One of the fundamental concepts in combinatorial theory is that of *enumeration*, and one of the basic techniques for dealing with problems of enumeration is that of *generating functions*. In this paper we shall survey some of the highlights of the theory of generating functions and shall discuss some applications to specific problems of enumeration. Many examples will be given—some classical and well-known, some more obscure, and a few new. Our object will be two-fold: (1) impart to the casual reader some of the flavor of recent work with generating functions, and (2) impart some facility for using generating functions as a tool for solving combinatorial problems. In some instances, such as Proposition 4.13, Proposition 5.3 and Example 6.11, we bring the reader near the frontiers of what we believe to be exciting new areas of research.

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Naturally, we can only give a small selection of topics from the vast subject of the enumerative theory of generating functions. In Section 3 we shall consider the "abstract" theory of generating functions. In Sections 4 and 5 we shall be concerned with two special classes of generating functions—rational functions and algebraic functions. Finally in Section 6 we shall discuss a result, known as the "exponential formula", which deals with the occurrence of the exponential function in certain types of enumeration problems. Further information about generating functions can be obtained from, e.g., [10], [28], [33]. These books are devoted almost entirely to the use of generating functions for solving combinatorial problems.

Let I be an index set, and let $\mathcal{S} = \{S_\iota : \iota \in I\}$ be a system of finite sets S_ι indexed by I. For our purposes, the *fundamental problem of enumeration* is to "determine" the cardinality of each S_ι as a function of $\iota \in I$. Equivalently, we wish to determine the *counting function* $N: I \to \mathbf{N}$ defined by $N(\iota) = |S_\iota|$. Here $\mathbf{N}$ denotes the set of nonnegative integers and $|S_\iota|$ denotes the cardinality (number of elements) of S_ι. In any combinatorial problem, there will be some combinatorial relationship between $\iota \in I$ and S_ι. For instance, we could have $I = \mathbf{N}$ with S_n being the set of all subsets of the set $[n] = \{1, 2, \ldots, n\}$. Here $N(n) = 2^n$. Examples of index sets frequently encountered in enumeration problems include the following:

(i) $\mathbf{N}$, the nonnegative integers.

(ii) $\mathbf{N} \times \mathbf{N}$, pairs (k, n) of nonnegative integers. For instance, $S_{(k,n)}$ could be all subsets of $[n]$ of cardinality k. Then $N(k, n)$ is commonly denoted $\binom{n}{k}$.

(iii) $\mathbf{P}$, the positive integers. For instance, S_n could be the set of divisors of n, so $N(n)$ is the well-known number-theoretic function $d(n)$.

(iv) π, the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of all nonnegative integers. Here $\lambda_i \in \mathbf{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots$, and $\Sigma\lambda_i$ is finite. If $\Sigma\lambda_i = n$, then λ is called a *partition of n*, denoted $\lambda \vdash n$. For instance, S_λ could be the set of all permutations in the symmetric group $\mathfrak{S}_n$ $(\lambda \vdash n)$ whose cycles have lengths $\lambda_1, \lambda_2, \ldots$. If we write $\lambda = < 1^{r_1} 2^{r_2} 3^{r_3} \cdots >$ to signify that exactly r_i of the λ_j's are equal to

i (so if $\lambda \vdash n$ then $\Sigma\, ir_i = n$), then $N(\lambda) = |S_\lambda| = n!/(1^{r_1}r_1!)(2^{r_2}r_2!)(3^{r_3}r_3) \cdots$.

II. GENERATING FUNCTIONS

We shall not attempt a rigorous general definition of generating functions but shall content ourselves with various examples. Heuristically, a generating function is a representation of a counting function $N: I \to \mathbf{N}$ as an element $F(N)$ of some algebra $\mathcal{A}$. The following are examples of types of generating functions which have actually arisen in specific enumeration problems.

2.1. Ordinary generating functions. Here $I = \mathbf{N}$, $\mathcal{A} = \mathbf{C}[[X]]$ (the ring of formal power series over the complex numbers $\mathbf{C}$), and $N: \mathbf{N} \to \mathbf{N}$ is represented by

$$F(N; X) = \sum_{n=0}^{\infty} N(n)\, X^n,$$

called the *ordinary generating function* of N. Sometimes $I = \mathbf{P}$ and the sum starts at $n = 1$.

2.2. Exponential generating functions. $I = \mathbf{N}$ and $\mathcal{A} = \mathbf{C}[[X]]$ as before, while

$$F(N; X) = \sum_{n=0}^{\infty} N(n)X^n/n!$$

2.3. Eulerian generating functions (cf. [16]). Let q be a fixed positive integer (almost always taken in practice to be a prime power corresponding to the field $GF(q)$). Take $I = \mathbf{N}$, $\mathcal{A} = \mathbf{C}[[X]]$, and

$$F(N; X) = \sum_{n=0}^{\infty} N(n)X^n/(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).$$

Frequently the denominator is replaced with $(1 - q)(1 - q^2) \cdots (1 - q^n)$; this amounts to the transformation $X \to X/(1 - q)$. One advantage of our "normalization" is that $F(N; X)$ reduces to an exponential generating function when $q = 1$.

2.4. Doubly-exponential generating functions. $I = \mathbf{N}$, $\mathcal{A} = \mathbf{C}[[X]]$, and

$$F(N; X) = \sum_{n=0}^{\infty} N(n)X^n/(n!)^2.$$

For instance [3], if $N(n)$ is the number of $n \times n$ matrices of nonnegative integers such that every row and column sum equals two, then $F(N; X) = e^{X/2}(1 - X)^{-1/2}$. (See Example 6.11.)

2.5. Chromatic generating functions (cf. [7], [32], [39]). $I = \mathbf{N}$, $\mathcal{A} = \mathbf{C}[[X]]$, $q \in \mathbf{P}$ is fixed, and

$$F(N; X) = \sum_{n=0}^{\infty} N(n)X^n/q^{\binom{n}{2}}\, n!.$$

Sometimes one sees $q^{\binom{n}{2}}$ replaced with $q^{n^2/2}$, amounting to the transformation $X \to X/q^{1/2}$.

2.6. Power series in two variables. Here $I = \mathbf{N} \times \mathbf{N}$ (or possibly $\mathbf{P} \times \mathbf{N}$, $\mathbf{N} \times \mathbf{P}$, $\mathbf{P} \times \mathbf{P}$) and $\mathcal{A} = \mathbf{C}[[X, Y]]$, the ring of formal power series in two variables X, Y over $\mathbf{C}$. Then $F(N; X, Y)$ can take such forms as

$$F(N; X, Y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N(m, n)X^m Y^n,$$

$$F(N; X, Y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N(m, n)X^m Y^n/n!,$$

$$F(N; X, Y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N(m, n) X^m Y^n / m! n!,$$

etc.

2.7. Power series in infinitely many variables. There are two common possibilities for I. One is the set $\mathbf{S}^*$ of all sequences $(n_1, n_2, \ldots)$ of nonnegative integers with only finitely many $n_i \neq 0$, while the other is the set π of all partitions of nonnegative integers. In either case $\mathfrak{A} = \mathbf{C}[[X_1, X_2, \ldots]] = \mathbf{C}[[\mathbf{X}]]$, the ring of formal power series in $X_1, X_2, \ldots$ over $\mathbf{C}$ (each monomial containing only finitely many different X_i). If $I = \mathbf{S}^*$, then

$$F(N; X_1, X_2, \ldots) = \sum_{n_1, n_2, \ldots = 0}^{\infty} N(n_1, n_2, \ldots) X_1^{n_1} X_2^{n_2} \cdots,$$

while if $I = \pi$, then

$$F(N; X_1, X_2, \ldots) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} N(\lambda) X_1^{r_1} X_2^{r_2} \cdots, \qquad (1)$$

where λ has exactly r_i parts equal to i.

2.8. Dirichlet series. $I = \mathbf{P}$ and $\mathfrak{A}$ is the algebra $\mathfrak{D}$ of all formal Dirichlet series with coefficients in $\mathbf{C}$. Then

$$F(N; s) = \sum_{n=1}^{\infty} N(n) n^{-s}.$$

As $\mathbf{C}$-algebras the two algebras $\mathfrak{D}$ and $\mathbf{C}[[\mathbf{X}]]$ are isomorphic (*via* the transformation $X_i \to p_i^{-s}$ where p_i is the ith prime), but of course their analytic behavior is entirely different.

III. BINOMIAL TYPE

The problem arises of trying to "explain" combinatorially why certain types of generating functions such as $\Sigma N(n) X^n$ and $\Sigma N(n) \times X^n/n!$ often arise, while other types like $\Sigma N(n) X^n/(1 + n^2)$ or $\Sigma N(n) X^n / 1^1 2^2 3^3 \cdots n^n$ never seem to occur. Two abstract theories of generating functions have been formulated to try to solve this problem—the Doubilet-Rota-Stanley theory of "reduced incidence algebras" [11], and the Bender-Goldman theory of "prefabs" [7] (cf. also the "dissect" theory of M. Henle [22], which combines features of both the preceding theories). To give the reader some feeling for this subject we shall discuss the main theorem of Doubilet-Rota-Stanley concerning power series generating functions in one variable.

A partially ordered set (or *poset*, for short) P will be said to be *binomial* if it satisfies the following three conditions:

(a) P is locally finite, i.e., every interval $[x, y] = \{z: x \leq z \leq y\}$ is finite, and P contains arbitrarily large finite chains. (A *chain* is a totally ordered subset of P.)

(b) For every interval $[x, y]$ of P, all maximal chains between x and y have the same length $n = n(x, y)$. We then call $[x, y]$ an n-*interval*. (The *length* of a chain is one less than its number of elements.)

(c) For all $n \in \mathbf{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains.

Clearly from these definitions we have $B(0) = B(1) = 1$, $B(2) = |[x, y]| - 2$, where $[x, y]$ is any 2-interval, and $B(0) \leq B(1) \leq B(2) \leq \cdots$.

Examples of binomial posets

3.1: $P = \mathbf{N}$ with the usual order. Then $B(n) = 1$ for all $n \in \mathbf{N}$.

3.2: P is the lattice of all finite subsets of $\mathbf{N}$, ordered by inclusion. Then $B(n) = n!$.

3.3: P is the lattice of all finite-dimensional subspaces of a vector space of infinite dimension over $GF(q)$, ordered by inclusion. Then $B(n) = (1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1})$.

3.4: P is the poset of all subsets of $\mathbf{N} \times \mathbf{N}$ of the form $S \times T$, where S and T are finite subsets of $\mathbf{N}$ of the same cardinality, ordered by inclusion. Then $B(n) = n!^2$.

3.5: Let V be an infinite vertex set, let $q \in \mathbf{P}$ be fixed, and let P be the set of all pairs (G, σ), where G is a function from all 2-sets

$\{u, v\} \subset V$ $(u \neq v)$ into $\{0, 1, \ldots, q-1\}$ such that all but finitely many values of G are 0, and where $\sigma: V \to \{0, 1\}$ is a map satisfying the two conditions: (a) if $G(\{u, v\}) > 0$ then $\sigma(u) \neq \sigma(v)$, and (b) $\sum_{v \in V} \sigma(v) < \infty$.

If (G, σ) and (H, τ) are in P, define $(G, \sigma) \leq (H, \tau)$ if:
(i) $\sigma(v) \leq \tau(v)$ for all $v \in V$, and
(ii) if $\sigma(u) = \tau(u)$ and $\sigma(v) = \tau(v)$, then $G(\{u, v\}) = H(\{u, v\})$.

Then P is a binomial poset with $B(n) = n!q^{\binom{n}{2}}$. This rather artificial-looking example arises naturally in [39, §3] in connection with the coloring of graphs.

Observe that the numbers $B(n)$ considered in 3.1–3.5 appear in the power series generating functions of 2.1–2.5. If we can somehow associate a binomial poset with generating functions of the form $\Sigma N(n)X^n/B(n)$, then we will have "explained" the form of the generating functions of 2.1–2.5. We also will have provided some justification of the vague metaprinciple that ordinary generating functions are associated with the nonnegative integers, exponential generating functions with sets, Eulerian generating functions with vector spaces, etc.

To see the connection between binomial posets and generating functions of the form $\Sigma N(n)X^n/B(n)$, it is necessary to consider incidence algebras. If P is any locally finite poset, the *incidence algebra* $I(P)$ of P (over $\mathbf{C}$, say) is the vector space of all functions $f: S(P) \to \mathbf{C}$, where $S(P)$ is the set of all nonvoid intervals $[x, y]$ of P, endowed with the multiplication (convolution)

$$fg(x, y) = \sum_{z \in [x, y]} f(x, z)g(z, y).$$

(We write $f(x, y)$ for $f([x, y])$, etc.) Note that the above sum is finite since P is locally finite. It is easily seen that $I(P)$ is an associative algebra with identity δ given by $\delta(x, y) = \delta_{xy}$ (the Kronecker delta). If P is binomial, let $R(P)$ be the subspace of $I(P)$ consisting of functions f constant on n-intervals, i.e., $f(x, y) = f(z, w)$ whenever $[x, y]$ and $[z, w]$ have the same length. If $f \in R(P)$, we write $f(n)$ for $f(x, y)$, where $[x, y]$ is an n-interval.

A fundamental property of binomial posets is that $R(P)$ is a sub-

algebra of $I(P)$, i.e., $R(P)$ is closed under convolution. Note also that $\delta \in R(P)$. Indeed, it is easy to see that

$$fg(n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} f(i)g(n-i), \tag{2}$$

where $\begin{bmatrix} n \\ i \end{bmatrix}$ denotes the number of elements z in an n-interval $[x, y]$ such that $[x, z]$ is an i-interval. Since $B(i)B(n-i)$ maximal chains of $[x, y]$ pass through a given such z, we have

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{B(n)}{B(i)B(n-i)}. \tag{3}$$

This is the P-analogue of the formula

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

This analogy is strengthened further by observing that

$$B(n) = A(n)A(n-1)\cdots A(1),$$

where $A(i) = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

We immediately have from (2) and (3) the following main theorem on binomial posets.

3.6. THEOREM: *Let P be a binomial poset. Then $R(P)$ is isomorphic to* $\mathbf{C}[[X]]$ *via*

$$f \mapsto F_f(X) = \sum_{n=0}^{\infty} f(n)X^n/B(n).$$

Let us consider some applications. In the following example, P is assumed to be a binomial poset:

3.7. Example: Define $\zeta \in R(P)$ by $\zeta(n) = 1$ for all $n \in \mathbf{N}$. Then for an n-interval $[x, y]$,

$$\zeta^2(n) = \zeta^2(x, y) = \sum_{z \in [x,y]} \zeta(x, z)\zeta(z, y)$$

$$= \sum_{z \in [x,y]} 1 = |[x, y]|.$$

Hence, the cardinality $N(n)$ of an n-interval is given by

$$\sum_{n=0}^{\infty} N(n)X^n/B(n) = (\sum_{n=0}^{\infty} X^n/B(n))^2.$$

Thus from 3.1 we have that the cardinality $N(n)$ of a chain of length n satisfies

$$\sum_{n=0}^{\infty} N(n)X^n = (\sum_{n=0}^{\infty} X^n)^2 = 1/(1 - X)^2 = \sum_{n=0}^{\infty} (n + 1)X^n,$$

whence $N(n) = n + 1$. Similarly from 3.2 the number $N(n)$ of subsets of an n-element set satisfies

$$\sum_{n=0}^{\infty} N(n)X^n/n! = (\sum_{n=0}^{\infty} X^n/n!)^2 = e^{2X} = \sum_{n=0}^{\infty} 2^n X^n/n!,$$

whence $N(n) = 2^n$. The analogous formula for Eulerian generating functions first appeared in [16].

3.8. Example: For $n \geq 1$, let $N(n)$ be the number of sequences $0 = a_0 < a_1 < \cdots < a_k = n$ of integers a_i such that no $a_{i+1} - a_i = 1$ for $0 \leq i < k$. Also set $N(0) = 1$. Let $P = \mathbf{N}$, and define $\eta \in R(P)$ by

$$\eta(n) = \begin{cases} 0, & n = 0 \text{ or } 1, \\ 1, & n \geq 2. \end{cases}$$

The number of sequences we seek of length k is clearly $\eta^k(n)$, so

$$N(n) = (\sum_{k=0}^{\infty} \eta^k)(n) = (1 - \eta)^{-1}(n).$$

Now $F_\eta(X) = X^2 + X^3 + \cdots = X^2/(1 - X)$, so

$$\sum_{n=0}^{\infty} N(n)X^n = (1 - F_\eta(X))^{-1}$$

$$= (1 - X)/(1 - X - X^2) \quad (4)$$

$$= 1 + X^2 + X^3 + 2X^4 + 3X^5 + \cdots.$$

It follows that $N(n + 1) = F_n$, the nth Fibonacci number, a well-known result. The reader should by now be able to find analogous results for sets, vector spaces, etc., and invent his own modifications and generalizations. For instance, if $|S| = n$ and $M(n)$ denotes the number of chains $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S$ such that each $|S_{i+1} - S_i| \geq 2$, $0 \leq i < k$, then in complete analogy to (4) we have

$$\sum_{n=0}^{\infty} M(n)X^n/n! = (1 - (e^X - 1 - X))^{-1} = (2 + X - e^X)^{-1}.$$

For a host of other applications and generalizations to other types of generating functions (such as Dirichlet series), see [11].

IV. RATIONAL FUNCTIONS IN ONE VARIABLE

Theorem 3.6 sheds considerable light on the "meaning" of generating functions and reduces certain simple types of combinatorial problems to a routine computation. However, it does not seem worthwhile to attack more complicated problems from this point of view. For the remainder of this paper we will consider other techniques for obtaining and analyzing generating functions. In this section we will consider some aspects of ordinary generating functions

$$F(X) = \sum_{0}^{\infty} N(n) X^n$$

which are rational functions in the ring $\mathbf{C}[[X]]$, i.e., for which there exist polynomials $P(X)$, $Q(X) \in \mathbf{C}[X]$ such that $F(X) = P(X)Q(X)^{-1}$. Here $Q(X)^{-1}$ is interpreted to be the element of $\mathbf{C}[[X]]$ satisfying $Q(X)Q(X)^{-1} = 1$. $Q(X)^{-1}$ will exist if and only if $Q(0) \neq 0$. The fundamental property of rational functions in $\mathbf{C}[[X]]$ from the viewpoint of enumeration is the following:

4.1. THEOREM: *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be a fixed sequence of complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. The following conditions on a function N: $\mathbf{N} \to \mathbf{C}$ are equivalent:*

(i) $\sum_{n=0}^{\infty} N(n)X^n = P(X)/Q(X)$,

where $Q(X) = 1 + \alpha_1 X + \alpha_2 X^2 + \cdots + \alpha_d X^d$ and $P(X)$ is a polynomial in X, of degree less than d, relatively prime to $Q(X)$.

(ii) *For all $n \geq 0$,*

$$N(n + d) + \alpha_1 N(n + d - 1) + \alpha_2 N(n + d - 2) + \cdots + \alpha_d N(n) = 0, \quad (5)$$

and N satisfies no relation $N(n + c) + \beta_1 N(n + c - 1) + \cdots + \beta_c N(n) = 0$, where $c < d$ and each β_i is a fixed element of $\mathbf{C}$.

(iii) *For all $n \geq 0$,*

$$N(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^{\,n},$$

where $1 + \alpha_1 X + \alpha_2 X^2 + \cdots + \alpha_d X^d = \prod_{i=1}^{k} (1 - \gamma_i X)^{d_i}$, the γ_i's are distinct, and $P_i(n)$ is a polynomial in n of degree $d_i - 1$. □

Theorem 4.1 is well known in the calculus of finite differences and has many proofs. Perhaps the simplest proof involves decomposing $P(X)/Q(X)$ by partial fractions. Other proofs can be given using the calculus of residues, finite difference operators, the method of undetermined coefficients, etc.

The main application of Theorem 4.1 is as follows: One frequently can show by non-combinatorial means that a generating function $\Sigma N(n)X^n$ is rational. Theorem 4.1(ii) and (iii) then provide a simple recurrence for calculating $N(n)$ and a means of estimating the growth of $N(n)$. We shall give as a non-trivial illustration a minor modification of a result of D. Klarner [23] and G. Pólya [46].

4.2. Example: A *polyomino* is a finite union P of unit squares in the plane such that the vertices of the squares have integer coordinates, and P is connected and has no finite cut set. Two polyominoes will be considered *equivalent* if there is a translation which transforms one into the other (reflections and rotations not allowed). Let $N(n)$ be the number of inequivalent n-square polyominoes P with the property that each "row" of P is an unbroken line of squares, i.e., if L is any line segment parallel to the x-axis with its two endpoints in P, then $L \subseteq P$. By convention set $N(0) = 0$. Then $N(1) = 1$, $N(2) = 2$, $N(3) = 6$, etc. It is easily seen that

$$N(n) = \Sigma(n_1 + n_2 - 1)(n_2 + n_3 - 1) \cdots (n_{s-1} + n_s - 1), \quad (6)$$

where the sum is over all ordered partitions $n_1 + n_2 + \cdots + n_s = n$ of n into positive integers n_i (by convention, the partition with $s = 1$ contributes 1 to the sum). Let $N_r(n)$ be the sum of those terms of (6) with $n_1 = r$, where we set $N_n(n) = 1$, and where we set $N_r(n) = 0$ if $r > n$ or $n < 0$. Thus

$$N(n) = \sum_{r=1}^{\infty} N_r(n),$$

$$N_r(n) = \sum_{i=1}^{\infty} (r + i - 1)N_i(n - r), \quad r < n. \quad (7)$$

Define the generating function

$$F(X, Y) = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} N_r(n) X^r Y^n,$$

so

$$F(1, Y) = \sum_{n=1}^{\infty} N(n) Y^n.$$

Now (7) implies

$$F(X, Y) = \sum_{n=1}^{\infty} X^n Y^n + \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} (r + i - 1) N_i(n - r) X^r Y^n =$$
$$\frac{XY}{1 - XY} + \frac{X^2 Y^2}{(1 - XY)^2} F(1, Y) + \frac{XY}{1 - XY} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} i N_i(n) Y^n, \tag{8}$$

by straightforward computation.

Let $\mathbf{D}$ be the subalgebra of $\mathbf{C}[[X, Y]]$ consisting of all power series $\sum_i \sum_j A_{ij} X^i Y^j$ such that for each $j \in \mathbf{N}$, only finitely many A_{ij} are unequal to 0. Define two linear operators $L_1, L_2\colon \mathbf{D} \to \mathbf{C}[[Y]]$ as follows:

$$L_1(\sum_i \sum_j A_{ij} X^i Y^j) = \sum_j (\sum_i A_{ij}) Y^j,$$

$$L_2(\sum_i \sum_j A_{ij} X^i Y^j) = \sum_j (\sum_i i A_{ij}) Y^j.$$

Note that L_1 and L_2 have the "representations"

$$L_1 H(X, Y) = H(1, Y), L_2 H(X, Y) = \frac{\partial}{\partial X} H(X, Y) \,|_{X=1};$$

however, for purposes of generalization it is convenient to regard L_1 and L_2 merely as "abstract" operators.

Define $G(Y) = L_2 F(X, Y)$. By applying L_1 and L_2 to (8), we obtain two linear equations involving $F(1, Y)$ and $G(Y)$. Specifically, we get:

$$F(1, Y) = \frac{Y}{1 - Y} + \frac{Y^2}{(1 - Y)^2} F(1, Y) + \frac{Y}{1 - Y} G(Y)$$
$$G(Y) = \frac{Y}{(1 - Y)^2} + \frac{2Y^2}{(1 - Y)^3} F(1, Y) + \frac{Y}{(1 - Y)^2} G(Y). \tag{9}$$

Here we have used the easily verified formulas

$$L_2\left(\frac{XY}{1 - XY}\right) = Y/(1 - Y)^2 \text{ and}$$
$$L_2\left(\frac{X^2 Y^2}{(1 - XY)^2}\right) = 2Y^2/(1 - Y)^3.$$

Eliminating $G(Y)$ from (9) allows us to solve for $F(1, Y)$ as a function of Y. The final result is

$$F(1, Y) = \frac{Y(1 - Y)^3}{1 - 5Y + 7Y^2 - 4Y^3}$$

$$= \frac{1}{16}\left(-5 + 4Y + \frac{5 - 13Y + 7Y^2}{1 - 5Y + 7Y^2 - 4Y^3}\right).$$

Hence we see that

$$N(n + 3) = 5N(n + 2) - 7N(n + 1) + 4N(n), \quad n \geq 2.$$

This recursion is by no means apparent, and no combinatorial proof of it is known.

It is evident that the above method (due essentially to D. Klarner [23], [24], who uses a certain integral representation of our operator L_2) will extend to a much wider class of problems. See also [46]. For instance, the above method yields after a tedious computation the following result:

4.3. PROPOSITION: *Define*

$$N(n) = \Sigma(f_{n_1} + f_{n_2} + f_{n_3})(f_{n_2} + f_{n_3} + f_{n_4}) \cdots (f_{n_s-2} + f_{n_s-1} + f_{n_s}),$$

where f is any function f: $\mathbf{P} \to \mathbf{C}$ *and where the sum is over all ordered partitions* $n_1 + n_2 + \cdots + n_s = n$ *of n* ($n_i \geq 1$). *By convention, a summand with* $s = 1$ *is 0 and with* $s = 2$ *is 1. Define*

$$F(X) = \sum_{n=1}^{\infty} N(n)X^n, f = \sum_{n=1}^{\infty} f_n X^n, \quad A = X/(1 - X).$$

Let $*$ *denote Hadamard product, i.e.,* $(\Sigma a_n X^n)*(\Sigma b_n X^n) = \Sigma a_n b_n X^n$. *Then*

$$F(X) = \frac{A^2}{(1-f)^2(1-f-f^2)-2A(f*f)(1-f^2)-A^2((f*f)^2+f*f*f)}. \square$$

In obtaining the above expression for $F(X)$ an enormous amount of cancellation takes place. This leads one to suspect that there is some simpler alternative method for obtaining such results. We do not, however, know of such a method.

Theorem 4.1 allows us to deduce the linear recurrence (5) which $N(n)$ satisfies from its generating function $P(X)/Q(X)$. We therefore ask what other properties of $N(n)$ can be "read off" from $P(X)/Q(X)$. A simple and elegant result along these lines has been given by Popoviciu [30] (cf. also [12], [41]). If we are given a function N: $\mathbf{N} \to \mathbf{C}$ satisfying a recurrence (5), then clearly there is a unique way of extending N to all of $\mathbf{Z}$ (the integers) such that (5) holds for all $n \in \mathbf{Z}$. Popoviciu's theorem relates the functions $N(n)$ and $N(-n)$. It is easily proved, e.g., by partial fractions.

4.4 THEOREM: *Let N*: $\mathbf{Z} \to \mathbf{C}$ *satisfy* (5) *for all* $n \in \mathbf{Z}$. Define

$$F(X) = \sum_{n=0}^{\infty} N(n)X^n, \quad \overline{F}(X) = \sum_{n=1}^{\infty} N(-n)X^n.$$

Then $F(X)$ *and* $\overline{F}(X)$ *are rational functions of X satisfying* $\overline{F}(X) = -F(1/X)$. $\square$

POLYNOMIALS

As important class of functions satisfying a recurrence (5) are the polynomials. In fact, we have the following corollary to Theorem 4.1:

4.5 COROLLARY: *The following conditions on a function N*: $\mathbf{N} \to \mathbf{C}$ *are equivalent*:

(i) $$\sum_{n=0}^{\infty} N(n)X^n = P(X)/(1 - X)^{d+1}, \tag{10}$$

where $P(X)$ *is a polynomial in X of degree at most d such that* $P(1) \neq 0$.

(ii) *For all* $n \geq 0$,

$$\sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} N(n+i) = 0,$$

while for some $n \geq 0$,

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} N(n+i) \neq 0.$$

(iii) $N(n)$ *is a polynomial in n of degree d.* $\square$

When a polynomial $N(n)$ arises combinatorially, frequently the coefficients of $P(X)$ (given by (10)) have a combinatorial significance. Moreover, Theorem 4.4 may give useful information about $P(X)$ *via* the following corollary:

4.6. COROLLARY: *Let N*: $\mathbf{Z} \to \mathbf{C}$ *be a polynomial of degree d,* *and let* $\sum_{n=0}^{\infty} N(n)X^n = P(X)/(1 - X)^{d+1}$, *where* $P(X) = a_0 + a_1X + \cdots + a_dX^d$.

(i) *Define r to be the greatest integer such that* $N(0) = N(1) = \cdots = N(r) = 0$. *(If* $N(0) \neq 0$, *let* $r = -1$.*) If* $r \neq -1$, *then r is the greatest integer such that* $a_0 = a_1 = \cdots = a_r = 0$. *Moreover,* $N(r+1) = a_{r+1}$ *whatever the value of r.*

(ii) *Define s to be the greatest integer such that* $N(-1) = N(-2) = \cdots = N(-s) = 0$. *(If* $N(-1) \neq 0$, let $s = 0$.*) If* $s \neq 0$, *then s is the greatest integer such that* $a_d = a_{d-1} = \cdots = a_{d-s+1} = 0$. *Moreover,* $N(-s-1) = (-1)^d a_{d-s}$ *whatever the value of s.*

(iii) *Let r and s be given by* (i) *and* (ii). *Then* $P(X) = X^{d+1+r-s} P(1/X)$ *if and only if* $N(n) = (-1)^d N(r - s - n)$ *for all* $n \in \mathbf{Z}$.

(iv) *The leading coefficient of* $N(n)$ is $P(1)/d!$.□

PARTITIONS AND PERMUTATIONS

The theory of partitions is a highly developed, elegant, and extensive branch of combinatorics. It originated with Euler in 1748 and has occupied the attention of many eminent researchers, such as Jacobi, Sylvester, Hardy and Littlewood, and MacMahon. For an introduction to this subject, see for example [20, Ch. 19], [4, Chs. 12-14], [1], [5], [44]. Generating functions have proved to be an invaluable tool in the study of partitions. We have space here to consider only a very small part of the subject, one in which rational generating functions play an important role. This is the subject of *P-partitions*, various aspects of which were considered by MacMahon, Bender, Knuth, Gordon, Kreweras, E. M. Wright, and others, with a general development first appearing in [37].

Let P be a finite partially ordered set of cardinality p. A *P-partition* of $n \in \mathbf{N}$ is an order-reversing map $\sigma: P \to \mathbf{N}$ satisfying $\sum_{x \in P} \sigma(x) = n$. The statement that σ is *order-reversing* means $\sigma(x) \geq \sigma(y)$ when $x \leq y$ in P. We say that σ is *strict* if $\sigma(x) > \sigma(y)$ when $x < y$ in P. If for instance P is a p-element chain, then a P-partition of n is equivalent to an ordinary partition of n into at most p parts, as defined in Section 1. If on the other extreme P is a disjoint union of p points, then a P-partition of n is equivalent to a *composition* (ordered partition) of n into p parts, allowing 0 as a part.

Define the following combinatorial concepts associated with P:

$a(n)$ = number of P-partitions of $n \in \mathbf{N}$.
$\bar{a}(n)$ = number of strict P-partitions of $n \in \mathbf{N}$.

$$F(X) = \sum_{n=0}^{\infty} a(n)X^n, \quad \bar{F}(X) = \sum_{n=0}^{\infty} \bar{a}(n)X^n.$$

$\Omega(m)$ = number of P-partitions $\sigma: P \to [m]$.
$\bar{\Omega}(m)$ = number of strict P-partitions $\sigma: P \to [m]$.
e_s = number of *surjective* P-partitions $P \to [s]$.
$\bar{e}_s$ = number of *surjective strict* P-partitions $P \to [s]$.

It is easily seen that if $p \geq 1$, then

$$\Omega(m) = \sum_{s=1}^{p} e_s \binom{m}{s}, \quad \bar{\Omega}(m) = \sum_{s=1}^{p} \bar{e}_s \binom{m}{s},$$

so $\Omega(m)$ and $\bar{\Omega}(m)$ are polynomials in m of degree p and leading coefficient $e_p/p!$.

We shall now establish the connection between P-partitions and permutations. Let $\omega: P \to [p]$ be a fixed *order-preserving bijection* (so $x \leq y$ in P implies $\omega(x) \leq \omega(y)$). Define the *JH-set* $\mathcal{L}$ of P to be the set of all permutations $\pi = (a_1, a_2, \ldots, a_p)$ of $(1, 2, \ldots, p)$ such that if $x < y$ in P, then $\omega(x)$ precedes $\omega(y)$ in π. (The reason for this terminology appears in [38].) Hence $\mathcal{L}$ contains a total of e_p permutations. If $\pi = (a_1, a_2, \ldots, a_p)$ is any permutation of $(1, 2, \ldots, p)$, a *descent* is a pair (a_i, a_{i+1}) such that $a_i > a_{i+1}$, while an *ascent* is such a pair with $a_i < a_{i+1}$. Let $\alpha(\pi)$ (respectively $\bar{\alpha}(\pi)$) be the number of descents (respectively, ascents) of the permutation π. Clearly $\alpha(\pi) + \bar{\alpha}(\pi) = p - 1$. The *greater index* $\iota(\pi)$ of π is defined by

$$\iota(\pi) = \Sigma\{j: a_j > a_{j+1}\}.$$

Similarly, the *lesser index* $\bar{\iota}(\pi)$ is defined by

$$\bar{\iota}(\pi) = \Sigma\{j: a_j < a_{j+1}\}.$$

Hence $\iota(\pi) + \bar{\iota}(\pi) = \binom{p}{2}$. (See [28, Section 104].)

We now state without proof some fundamental results concerning P-partitions. Proofs of more general results may be found in [37], especially Corollary 7.2 and Proposition 13.3.

4.7. PROPOSITION: (i) *$F(X)$ and $\overline{F}(X)$ are rational functions of X given explicitly by*

$$F(X) = \Big(\sum_{\pi\in\mathcal{L}} X^{\iota(\pi)}\Big)/(1-X)(1-X^2)\cdots(1-X^p).$$

$$\overline{F}(X) = \Big(\sum_{\pi\in\mathcal{L}} X^{\bar{\iota}(\pi)}\Big)/(1-X)(1-X^2)\cdots(1-X^p).$$

(ii) *We have*

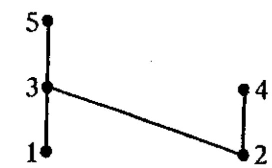
$$\sum_{m=0}^{\infty} \Omega(m)X^m = \Big(X\cdot \sum_{\pi\in\mathcal{L}} X^{\alpha(\pi)}\Big)/(1-X)^{p+1},$$

$$\sum_{m=0}^{\infty} \overline{\Omega}(m)X^m = \Big(X\cdot \sum_{\pi\in\mathcal{L}} X^{\bar{\alpha}(\pi)}\Big)/(1-X)^{p+1}.\square$$

Using the formulas $\alpha(\pi) + \bar{\alpha}(\pi) = p - 1$, $\iota(\pi) + \bar{\iota}(\pi) = \binom{p}{2}$, Theorem 4.4, and the definition of $\mathcal{L}$, we can obtain many interesting corollaries to Proposition 4.7, a sample of which are contained in the following:

4.8. COROLLARY:
(i) *$X^p\overline{F}(X) = (-1)^pF(1/X)$.*
(ii) *$\overline{\Omega}(m) = (-1)^p\Omega(-m)$.*
(iii) *Let $F(X) = W(X)/(1-X)(1-X^2)\cdots(1-X^p)$. Then $W(X)$ is a monic polynomial with nonnegative integer coefficients of degree $\binom{p}{2} - \sum_{x\in P}\delta(x)$, where $\delta(x)$ is the length of the longest chain of P with bottom x. Moreover, $W(0) = 1$ and $W(1) = e_p$.*
(iv) *Let $d = \deg W(X)$. Then $W(X) = X^dW(1/X)$ if and only if for each $x \in P$, every maximal chain of the sub-partially ordered set $\{y: y \geq x\}$ has the same length $l = l(x)$.*

4.9. Example: Let (P, ω) be given by



Then $\mathcal{L}$ is given by:

π					$\alpha(\pi)$	$\bar{\alpha}(\pi)$	$\iota(\pi)$	$\bar{\iota}(\pi)$
1	2	3	4	5	0	4	0	10
2	1	3	4	5	1	3	1	9
1	2	4	3	5	1	3	3	7
1	2	3	5	4	1	3	4	6
2	4	1	3	5	1	3	2	8
2	1	4	3	5	2	2	4	6
2	1	3	5	4	2	2	5	5

Hence

$$F(X) = (1 + X + X^2 + X^3 + 2X^4 + X^5)/ (1-X)(1-X^2)\cdots(1-X^5),$$

$$\overline{F}(X) = (X^5 + 2X^6 + X^7 + X^8 + X^9 + X^{10})/ (1-X)(1-X^2)\cdots(1-X^5),$$

$$\sum_{m=0}^{\infty} \Omega(m)X^m = (X + 4X^2 + 2X^3)/(1-X)^6,$$

$$\sum_{m=0}^{\infty} \overline{\Omega}(m)X^m = (2X^3 + 4X^4 + X^5)/(1-X)^6.$$

4.10. Example: Suppose P is a disjoint union of p points. Then it can be seen directly that

$$\Omega(m) = \overline{\Omega}(m) = m^p,$$

$$F(X) = \overline{F}(X) = 1/(1 - X)^p.$$

Moreover, $\mathfrak{L}$ consists of *all* $p!$ permutations of $[p]$, and Proposition 5.3 reduces to classical results on permutations. For instance, the total number of permutations of $[p]$ with s descents is known as an *Eulerian number*, denoted by Knuth [26, Vol. 3, 5.1.3] as $\left\langle {p \atop s+1} \right\rangle$. Proposition 4.7 implies the well-known result (e.g., [33, pp. 38–39], [26, Vol. 3, 5.1.3, Eq. 8], [10, Ch. 6.5])

$$\sum_{m=0}^{\infty} m^p X^m = \left(\sum_{s=1}^{p} \left\langle {p \atop s} \right\rangle X^s\right) / (1 - X)^{p+1}.$$

Similarly Proposition 4.7 implies that

$$\sum_{\pi} X^{i(\pi)} = (1 + X)(1 + X + X^2) \cdots (1 + X + X^2 + \cdots + X^{p-1}),$$

where the sum is over all permutations π of $[p]$. This remarkable formula is due to MacMahon [27] [26, Vol. 3, 5.1.1].

4.11. Example: Let $P = C_p$, a p-element chain. Then $\mathfrak{L}$ consists of the single permutation $(1, 2, \ldots, p)$, and

$$\Omega(m) = \binom{m + p - 1}{p}, \quad \overline{\Omega}(m) = \binom{m}{p},$$

$$F(X) = 1/(1 - X)(1 - X^2) \cdots (1 - X^p),$$

$$\overline{F}(X) = X^{\binom{p}{2}}/(1 - X)(1 - X^2) \cdots (1 - X^p).$$

The formulas for $\Omega(m)$ and $\overline{\Omega}(m)$ are simply the fundamental expressions for counting combinations with or without repetition, while the formulas for $F(X)$ and $\overline{F}(X)$ are basic identities in the theory of partitions.

4.12. Example: Let $P = C_r \times C_s$, a direct (cartesian) product of two chains of cardinalities r and s, where say $r \le s$. It is by no means *a priori* evident that explicit expressions can be given for $\Omega(m)$ and $F(X)$, but such is indeed the case. Namely,

$$\Omega(m) = \frac{\binom{r+m-1}{r}\binom{r+m}{r}\binom{r+m+1}{r}\cdots\binom{r+m+s-2}{r}}{\binom{r}{r}\binom{r+1}{r}\binom{r+2}{r}\cdots\binom{r+s-1}{r}}$$

$$F(X) = 1/(1)(2)^2(3)^3 \cdots (r)^r(r+1)^r \cdots (s)^r(s+1)^{r-1}(s+2)^{r-2} \cdots (r+s-1)^1,$$

where $(\mathbf{k}) = 1 - x^k$. These remarkable formulas belong to the fascinating subject of *plane partitions* and are intimately connected with symmetric functions and the representation theory of the symmetric group. For further information, see [36].

The myriad possibilities for modifying or extending the theory of P-partitions remains largely unexplored. As a modest example of what can be done in this direction, we state without proof the following recent result [45].

Let Q_p be the set of all sequences $\pi = (a_1, a_2, \ldots, a_{2p})$ such that each integer $i \in [p]$ appears exactly twice, and such that if $i < j < k$ and $a_i = a_k$, then $a_j > a_i$. It is easily seen that Q_p has cardinality $1 \cdot 3 \cdot 5 \cdots (2p - 1)$. A *descent* of $\pi \in Q_p$ is a pair (a_i, a_{i+1}) with $a_i > a_{i+1}$ $(1 \le i \le 2p - 1)$. Let $s(n, k)$ and $S(n, k)$ denote the Stirling numbers of first and second kinds, respectively. (For a discussion of these numbers, see for example [10, Ch. V].)

4.13. PROPOSITION: *We have the identities*

$$\sum_{n=1}^{\infty} S(n+p, n)X^n = \left(\sum_{i=1}^{p} B_{p,i}X^i\right)/(1-X)^{2p+1}$$

and

$$(-1)^p \sum_{n=1}^{\infty} s(n+p, n)X^n = \left(\sum_{i=1}^{p} B_{p-i+1,i}X^i\right)/(1-X)^{2p+1},$$

where $B_{p,i}$ is equal to the number of sequences $\pi \in B_{p,i}$ with exactly $i-1$ descents. □

For instance, if $p = 2$ then we have (omitting superfluous parentheses and commas) $Q_2 = \{1122, 1221, 2211\}$. Hence

$$\sum_{n=1}^{\infty} S(n+2, n)X^n = (X + 2X^2)/(1-X)^5$$

and

$$\sum_{n=1}^{\infty} s(n+2, n)X^n = (2X + X^2)/(1-X)^5,$$

agreeing with the known results

$$S(n+2, n) = \binom{n+3}{4} + 2\binom{n+2}{4}$$

and

$$s(n+2, n) = 2\binom{n+3}{4} + \binom{n+2}{4}.$$

V. ALGEBRAIC FUNCTIONS

The elements $F(X)$ of $\mathbf{C}[[X]]$ of the next "level of complexity"

after the rational functions are the *algebraic functions*. By definition, $Y = F(X)$ is an algebraic function (over $\mathbf{C}$) if there exist polynomials $P_0(X), P_1(X), \ldots, P_d(X) \in \mathbf{C}[X]$ such that

$$P_0(X) + P_1(X)Y + \cdots + P_d(X)Y^d = 0, \tag{11}$$

as an element of $\mathbf{C}[[X]]$. The least possible d for which (11) holds is the *degree* of Y. If Y satisfies (11), then Y has degree d if and only if $P_d(X) \neq 0$ and $P_0(X) + P_1(X)Y + \cdots + P_d(X)Y^d$ is *irreducible*, considered as a polynomial in Y over the field $\mathbf{C}(X)$.

The theory of algebraic functions has been extensively developed, but most of the results have no direct application to problems of enumeration. We shall discuss some results which *do* apply to enumeration and give several examples to indicate how algebraic functions actually arise in enumeration problems.

5.1. THEOREM (Comtet [9]): *Let $Y = F(X)$ be an algebraic function of degree d, given by $F(X) = \sum_{n=0}^{\infty} N(n)X^n$. Then there exists a positive integer q and polynomials $p_0(n), p_1(n), \ldots, p_q(n)$, such that $p_q(n) \neq 0$, $\deg p_i(n) < d$, and for all n sufficiently large,*

$$p_q(n)N(n+q) + p_{q-1}(n)N(n+q-1) + \cdots + p_0(n)N(n) = 0. \tag{12}$$

Sketch of proof (a streamlined version of Comtet's proof):

Let Y satisfy (11). By differentiating (11) repeatedly with respect to X and using induction, we get that $Y^{(k)} = d^kY/dX^k$ is a rational function $R_k(X, Y)$ of X and Y for all $k \geq 0$. Since Y is algebraic of degree d over $\mathbf{C}(X)$, the functions $1, Y^{(0)} = Y, Y^{(1)}, \ldots, Y^{(d-1)}$ are linearly dependent over $\mathbf{C}(X)$. Write this dependence relation, clear denominators so the coefficient of each $Y^{(k)}$ is a polynomial in X, expand each $Y^{(k)}$ as a power series in X, and equate coefficients of X^n on both sides of the dependence relation to get the desired result. □

5.2. Example: Suppose $2Y^2 - (1 + X)Y + X = 0$, where $Y = \sum_{n=0}^{\infty} N(n)X^n$. Then $Y' = (Y - 1)/(4Y - 1 - X)$ and

$$Y'(X^2 - 6X + 1) - (X - 3)Y + (X - 1) = 0,$$

from which we get

$$(n + 2)N(n + 2) - 3(2n + 1)N(n + 1) + (n - 1)N(n) = 0, \; n \geq 1.$$

5.3. THEOREM: *Let* $F(Y, Z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N(m, n)Y^m Z^n \in \mathbf{C}[[Y, Z]]$. *The diagonal* $D_F(X)$ *of* $F(Y, Z)$ *is the series* $\sum_{n=0}^{\infty} N(n, n)X^n \in \mathbf{C}[[X]]$. *If* $F(Y, Z)$ *is a rational function of* Y *and* Z, *then* $D_F(X)$ *is an algebraic function of* X.

Sketch of proof: Regard $F(Y, Z)$ and $D_F(X)$ as functions of the complex variables X, Y, Z. It is easily seen that $F(Y, Z)$ and $D_F(X)$ converge for X, Y, Z sufficiently small in absolute value (when F is rational). Let C be any sufficiently small circle about the origin in the complex s-plane. We then have for all X sufficiently small (depending on C) in absolute value,

$$D_F(X) = \frac{1}{2\pi i} \int_C F(s, X/s) \frac{ds}{s}.$$

(Cf. [25, Theorem 1] for a rigorous justification of this "formal identity".) By the residue theorem, $D_F(X)$ is equal to the sum of the residues at those poles $s = s(X)$ satisfying $s \to 0$ as $X \to 0$. To compute these residues, write $F(s, X/s) = P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials in s with coefficients in $\mathbf{C}[X]$. Then the roots of $Q(s)$ are algebraic functions of X. Thus the poles of $F(s, X/s)$ are algebraic functions of X, so the residues at these poles will be rational combinations of these algebraic functions and hence themselves algebraic. From this the proof follows.□

5.4. Example: Let S be a subset of $\mathbf{N} \times \mathbf{N}$ such that $(0, 0) \notin S$. Let $N_S(m, n)$ be the number of ways the vector (m, n) can be written as a sum of vectors belonging to S. The order of summands is taken into account, so for example if

$$S = \{(1, 0), (1, 1), (0, 1)\},$$

then $N_S(1, 1) = 3$, corresponding to $(1, 0) + (0, 1)$, $(0, 1) + (1, 0)$, and $(1, 1)$. Define

$$F_S(Y, Z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} N_S(m, n)Y^m Z^n.$$

It is easily seen that

$$F_S(Y, Z) = 1\Big/\left(1 - \sum_{(i,j)\in S} Y^i Z^j\right).$$

Hence if S is finite, $F_S(Y, Z)$ is a rational function of Y and Z, so from Theorem 5.3 we obtain:

5.5. THEOREM: *Let* S *be a finite subset of* $\mathbf{N} \times \mathbf{N}$ *with* $(0, 0) \notin S$. *Define*

$$G_S(X) = \sum_{n=0}^{\infty} N_S(n, n)X^n.$$

Then $G_S(X)$ *is an algebraic function of* X, *and hence* (by Theorem 5.1) $N(n, n)$ *satisfies a recursion of the form* (12) *for* n *sufficiently large.*□

For an explicit example, take $S = \{(0, 1), (1, 0), (1, 1)\}$. Then $F_S(Y, Z) = 1/(1 - Y - Z - YZ)$ and

$$G_S(X) = \frac{1}{2\pi i} \oint \frac{ds}{s\left(1 - s - \frac{X}{s} - X\right)}$$

$$= -\frac{1}{2\pi i} \oint \frac{ds}{s^2 + (X - 1)s + X}.$$

The only pole s of the integrand satisfying $s \to 0$ as $X \to 0$ occurs for

$$s = \frac{1 - X - \sqrt{1 - 6X + X^2}}{2}$$

The residue at this pole is

$$G_S(X) = (1 - 6X + X^2)^{-1/2}.$$

For further information on this special case, see [10, Ch. 1, Ex. 21].

It is not necessary for S to be finite in order for $F_S(Y, Z)$ to be rational. For instance, if $S = \mathbf{N} \times \mathbf{N} - \{(0, 0)\}$, then

$$F_S(Y, Z) = 1/\left(1 - \left(\frac{1}{(1-Y)(1-Z)} - 1\right)\right),$$

and we obtain

$$G_S(X) = \sum_{n=0}^{\infty} N_S(n, n)X^n = \tfrac{1}{2}[1 + (1 - 12X + 4X^2)^{-1/2}].$$

There is one remaining theorem concerning algebraic functions which is useful in enumeration problems. This is the *Lagrange inversion formula.* Lagrange's formula allows in certain cases an explicit determination of the coefficients of a power series defined by a functional equation, such as an algebraic function. Lagrange's formula is normally stated for analytic functions (e.g., [43, p. 132]), but we shall state a special case valid for formal power series (cf., e.g., [31, Section 5]). We shall use notation from the calculus in a formal way. For instance, if $F(X) = \sum_{n=0}^{\infty} N(n)X^n$, then

$$F(0) = N(0),\ F'(0) = N(1),$$

$$d^2F/dX^2 = \sum_{n=0}^{\infty} (n + 1)(n + 2)N(n + 2)X^n, \text{ etc.}$$

5.6. THEOREM: *Let* $\Phi(X) \in \mathbf{C}[[X]]$. *There is a unique* $Y = Y(X) \in \mathbf{C}[[X]]$ *such that*

$$Y = X \cdot \Phi(Y). \tag{13}$$

(Note that the computation of $\Phi(Y)$ does not involve questions of convergence, since from (13) $Y(0) = 0$ and therefore the coefficient of X^n in the expansion of $\Phi(Y)$ is given by a finite sum.) *Let* $F(X) \in \mathbf{C}[[X]]$. *Then*

$$F(Y) = F(0) + \sum_{n=1}^{\infty} \frac{a_n X^n}{n!},$$

where

$$a_n = \frac{d^{n-1}}{dX^{n-1}} [F'(X)\phi(X)^n]_{X=0}. \square$$

5.7. Example: Let $Y = X + Y^2 + Y^3$, with $Y(0) = 0$. Thus $Y = X/(1 - Y - Y^2)$. If we let $\Phi(X) = 1/(1 - X - X^2)$ and $F(X) = X$, then we have

$$Y = \sum_{n=1}^{\infty} \frac{X^n}{n!} \left[\frac{d^{n-1}}{dX^{n-1}} (1 - X - X^2)^{-n}\right]_{X=0}.$$

Now, $(1 - X - X^2)^{-n} = \sum_{n=0}^{\infty} \binom{-n}{k} (-1)^k (X + X^2)^k$. Hence if $Y = \sum_{n=1}^{\infty} N(n)X^n$, then we easily obtain

$$N(n) = \sum_{\substack{a+2b=n-1 \\ a,\, b \in \mathbf{N}}} \frac{(n + a + b - 1)!}{n!\,a!\,b!}. \tag{14}$$

We conclude this section with an example which is a prototype for a wide class of results dealing with planar maps, parenthesization, trees, formal languages, and related topics.

5.8. Example: Let $S \subseteq \mathbf{P} - \{1, 2\}$. Let $N_S(n)$ be the number of ways of dividing a convex $(n+1)$-gon C into regions R by drawing diagonals not intersecting in the interior of C, such that the number of sides of each region R belongs to S. By convention, $N(0) = 0$, $N(1) = 1$. Fix an edge E of C. Given a decomposition of C enumerated by $N_S(n)$, let k be the number of edges of the region containing E. If we remove E from C, we obtain $k - 1$ new decomposed polygons $C_1, \ldots, C_{k-1}$. If $e(K)$ denotes one less than the number of edges of a polygon K, then $e(C) = e(C_1) + \cdots + e(C_{k-1})$. Hence

$$N_S(n) = \sum_{k \in S} \sum_{\substack{b_1 + \cdots + b_{k-1} = n \\ b_i \in \mathbf{N}}} N_S(b_1) \cdots N_S(b_{k-1}), \quad n \geq 2. \tag{15}$$

Let $Y = F_S(X) = \sum_{n=1}^{\infty} N_S(n) X^n$. Then (15) yields

$$Y = X + \sum_{k \in S} Y^{k-1}. \tag{16}$$

Under certain circumstances Y will be algebraic, e.g., when S is finite, and previous results of this section can be applied.

Suppose, for instance, $S = \{k\}$, $k \geq 3$. Thus, $Y = X + Y^{k-1}$. Theorem 5.6 can be used to show

$$N_S(n) = \begin{cases} 0, \text{ if } (k-2) \nmid (n-1), \\ \dfrac{1}{n+t}\dbinom{n+t}{t}, \text{ if } n - 1 = (k-2)t. \end{cases}$$

The simple form of this answer suggests that a combinatorial proof might be possible. The expression

$$\frac{1}{n+t}\binom{n+t}{t}$$

is equal to the number of circular permutations of n red beads and t white beads, since $(n, t) = 1$. Thus, we seek an explicit one-to-one correspondence between these circular permutations and the appropriate divisions of a polygon. Such one-to-one correspondences have been described by Raney [31], Tamari [42], and others. These authors prove more generally that the number $L(n)$ of ways of dividing an $(n+1)$-gon with d diagonals to form a_i regions with $i + 1$ sides (so $d + 1 = \Sigma a_i$ and $n - 1 = \Sigma(i-1)a_i$) is equal to

$$L(n) = (n+d)!/n!\ \Pi(a_i!). \tag{17}$$

This result was first proved by Etherington and Erdelyi [13] using generating functions. If for example we take $a_2 = a$, $a_3 = b$, and all other $a_i = 0$, we can deduce (14) purely combinatorially. More generally, Raney [31] has shown that (17) is sufficiently general to yield a purely combinatorial proof of Theorem 5.6.

The function Y of (16) can be algebraic without S being finite. For instance, take $S = \{3, 4, 5, \ldots\}$, so $N_S(n)$ is equal to the *total* number of ways of dividing an $(n+1)$-gon by diagonals not intersecting in the interior. This is the "second problem of Schröder" [35]. By (16), we have

$$Y = X + \sum_{k=3}^{\infty} Y^{k-1} = X + \frac{Y^2}{1-Y},$$

so $2Y^2 - (1+X)Y + X = 0$. This gives

$$Y = \frac{1}{4}\left(1 + X - (1 - 6X + X^2)^{1/2}\right).$$

This power series Y was the one considered in Example 5.2, so we get

$$(n+2)N_S(n+2) - 3(2n+1)N_S(n+1) + (n-1)N_S(n) = 0, \quad n \geq 1,$$

as first observed by Comtet [10, p. 57]. (Comtet's formula has a misprint.)

For excellent bibliographies of the many variations of Example 5.8, see [2], [8], or [17]. For the problem of asymptotically estimating the coefficients of an algebraic function, and of asymptotic estimates in general as applied to enumeration, see for example [6]. Finally, we mention that a simpler approach than ours for handling certain types of algebraic functions appears in [26, Vol. 1, Section 2.2.1, Exercises 4 and 11], especially pages 532-534.

VI. THE EXPONENTIAL FORMULA

We wish to explain the ubiquitous appearance in combinatorial enumeration problems of the exponential function. In Section 3 we saw that the exponential function is associated with the incidence algebra of the lattice of finite subsets of **N**; however, there are many occurrences of the exponential function in combinatorics which cannot be explained in this manner. We will present a general result (Corollary 6.2), which we call the "exponential formula for r-partitions", which leads to a plethora of generating functions involving the exponential function. Although an even more general exponential formula can be given, for simplicity's sake we will restrict ourselves to r-partitions. There are many different approaches to deriving the exponential formula; we choose one which seems to involve the least preparation. A wide variety of examples and special cases will be discussed.

Let S be a finite set with n elements. Recall that a *partition* of S is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of non-empty pairwise disjoint subsets B_i of S whose union is S. We say that π is of *type* $(a_1, a_2, \ldots, a_n)$ if exactly a_i of the B_j's have i elements. Thus $\Sigma ia_i = n$ and $\Sigma a_i = k$. We call the subsets B_i the *blocks* of π and say that π has k blocks, denoted $|\pi| = k$. The number of partitions of S with k blocks is the Stirling number $S(n, k)$ of the second kind, while the total number of partitions of S is the Bell number $B(n)$ (see, e.g., [10, Ch. V]). Let Π_n denote the set of all partitions of $[n] = \{1, 2, \ldots, n\}$.

More generally, if r is a fixed positive integer and if S is an n-element set, define an *r-partition of S* to be a set

$$\pi = \{(B_{11}, B_{12}, \ldots, B_{1r}), (B_{21}, B_{22}, \ldots, B_{2r}), \ldots, (B_{k1}, B_{k2}, \ldots, B_{kr})\}$$

satisfying:

(i) For each $j \in [r]$, the set $\pi_j = \{B_{1j}, B_{2j}, \ldots, B_{kj}\}$ forms a

partition of S into k blocks. Thus each B_{ij} is a nonvoid subset of S, the k sets $B_{1j}, \ldots, B_{kj}$ are pairwise disjoint, and $\bigcup_i B_{ij} = S$.

(ii) For fixed i, $|B_{i1}| = |B_{i2}| = \cdots = |B_{ir}|$.

It follows that the r partitions $\pi_1, \ldots, \pi_r$ all have the same type $(a_1, a_2, \ldots, a_n)$, which we call the *type* of π. We also say that π has k blocks, denoted $|\pi| = k$. We let Π_{nr} denote the set of all r-partitions of $[n]$, so $\Pi_{n1} = \Pi_n$.

6.1. THEOREM (the convolutional formula for r-partitions): *Let f: $\mathbf{P} \to \mathbf{C}$ and g: $\mathbf{P} \to \mathbf{C}$. Define a new function h: $\mathbf{P} \to \mathbf{C}$ by*

$$h(n) = \sum_\pi f(1)^{a_1} f(2)^{a_2} \cdots f(n)^{a_n} g(|\pi|),$$

where π ranges over all r-partitions of $[n]$, and where π has type $(a_1, a_2, \ldots, a_n)$. Define the power series $F(X)$, $G(X)$, $H(X) \in \mathbf{C}[[X]]$ by

$$F(X) = \sum_1^\infty f(n)X^n/n!^r, \quad G(X) = \sum_1^\infty g(n)X^n/n!,$$

$$H(X) = \sum_1^\infty h(n)X^n/n!^r.$$

Then $H(X) = G(F(X))$.

Proof: We have

$$G(F(X)) = \sum_{k=1}^\infty g(k)\left[\sum_{i=1}^\infty f(i)X^i/i!^r\right]^k/k!$$

$$= \sum_{k=1}^\infty \frac{g(k)}{k!} \Sigma \frac{f(b_1)f(b_2)\cdots f(b_k)}{b_1!^r b_2!^r \cdots b_k!^r} X^{b_1+b_2+\cdots+b_k},$$

where the inner sum is over all k-tuples $(b_1, \ldots, b_k) \in \mathbf{P}^k$. Let a_i be the number of b_j's which are equal to i, so that $k = \Sigma a_i$; and let $n = \Sigma b_i = \Sigma ia_i$. We obtain

$$G(F(X)) = \sum_{n=1}^{\infty} \frac{X^n}{n!^r} \sum \frac{n!^r\, \alpha(a_1, \ldots, a_n)}{(1!^{a_1}\, 2!^{a_2} \cdots n!^{a_n})^r k!} f(1)^{a_1} \cdots f(n)^{a_n} g(k),$$

where the inner sum is over all solutions in non-negative integers a_i to $n = \Sigma i a_i$, where $k = \Sigma a_i$, and where $\alpha(a_1, \ldots, a_n)$ is the number of distinct k-tuples $(b_1, \ldots, b_k)$ with exactly a_i of the b_j's equal to i. Clearly $\alpha(a_1, \ldots, a_n)$ is the multinomial coefficient $k!/a_1!a_2! \cdots a_n!$. Hence

$$G(F(X)) =$$

$$\sum_{n=1}^{\infty} \frac{X^n}{n!^r} \sum \frac{n!^r}{(1!^{a_1} \cdots n!^{a_n})^r\, a_1! \cdots a_n!} f(1)^{a_1} \cdots f(n)^{a_n} g(k).$$

It is easily proved that the number of r-partitions of $[n]$ of type $(a_1, \ldots, a_n)$ is just $n!^r/(1!^{a_1} \cdots n!^{a_n})^r\, a_1! \cdots a_n!$. From this the proof follows.□

6.2. Corollary (the exponential formula for r-partitions): *In Theorem* 6.1, *let* $g(n) = 1$ *for all* $n \in \mathbf{P}$. *Then*

$$1 + H(X) = \exp F(X).\square$$

More sophisticated approaches to Theorem 6.1 and Corollary 6.2 are given in [15, Ch. III], [14], [11, Thm. 5.1], and [7, §3]. The first three of these references treat only the case $r = 1$, and our viewpoint most closely follows [11]. The prefab theory of [7] gives more general results than Corollary 6.2, though it is possible to extend Theorem 6.1, in a manner analogous to our treatment of binomial posets in Section 3, so that it achieves the same level of generality as the treatment in [7].

We conclude this section with a number of applications of Theorem 6.1 and Corollary 6.2.

6.3. Example: If we set $f(n) = Y$ for all $n \in \mathbf{P}$ in Corollary 6.2, then $h(n) = \sum_k S_r(n, k)Y^k$, where $S_r(n, k)$ is the number of r-partitions of $[n]$ into k blocks. Hence from Corollary 6.2 we obtain

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} S_r(n, k) X^n Y^k / n!^r = \exp Y \sum_{n=1}^{\infty} X^n / n!^r.$$

In particular (putting $Y = 1$),

$$1 + \sum_{1}^{\infty} |\Pi_{nr}| \cdot X^n/n!^r = \exp \sum_{n=1}^{\infty} X^n/n!^r.$$

When $r = 1$, $S_r(n, k)$ becomes the Stirling number $S(n, k)$ of the second kind, and $|\Pi_{nr}|$ becomes the Bell number $B(n)$. We recover the well-known results (see, e.g., [10, Ch. V])

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} S(n, k) X^n Y^k / n! = \exp Y(e^X - 1),$$

$$1 + \sum_{n=1}^{\infty} B(n) X^n / n! = \exp (e^X - 1).$$

6.4. Example: Let $f(n)$ be the number of connected graphs (without loops or multiple edges) on the vertex set $[n]$, and let $h(n, k)$ be the total number of graphs on $[n]$ with k connected components. A graph with k components can be obtained uniquely by partitioning $[n]$ into k blocks and "attaching" a connected graph to each block. If a block B has i elements, then there are $f(i)$ connected graphs which can be placed on B. Hence

$$\sum_k h(n, k) Y^k = \sum_{\pi \in \Pi_n} f(1)^{a_1} \cdots f(n)^{a_n}\, Y^{a_1 + \cdots + a_n}$$

where in the latter sum π has type $(a_1, a_2, \ldots, a_n)$. From Corollary 6.2 (with $r = 1$) we obtain

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h(n, k) X^n Y^k / n! = \exp Y \cdot \sum_{1}^{\infty} f(n) X^n / n!.$$

Clearly there are a total of $2^{\binom{n}{2}}$ graphs on the vertex set $[n]$. Hence setting $Y = 1$ in the above formula, we get

$$\sum_{n=0}^{\infty} 2^{\binom{n}{2}} X^n/n! = \exp \sum_{n=1}^{\infty} f(n)X^n/n!. \tag{18}$$

Note that the above power series has zero radius of convergence, but this need not be a cause of concern since (18) is a *formal* power series identity.

Example 6.4 is the archetypal application of the exponential formula in the case $r = 1$. Whenever we have some structure on $[n]$ which is "pieced together" from its connected components, we obtain a formula analogous to (18). For instance, instead of graphs we could equally well have used partial orders, topologies, diagraphs, etc.

6.5. Example: Let $h(n)$ be the number of graphs G on the vertex set $[n]$, such that every component of G is a cycle (of length ≥ 3), an edge, or a single vertex. There are $(i-1)!/2$ cycles on an i-element set, $i \geq 3$. (Of course we mean an undirected cycle; there are $(i-1)!$ directed cycles.) Moreover, there is one two-vertex graph with an edge, and only one single-vertex graph. Hence from Corollary 6.2 (with $r = 1$) we get

$$1 + \sum_1^{\infty} h(n)X^n/n! = \exp\left[X + \frac{X^2}{2} + \frac{1}{2}\sum_3^{\infty}\frac{X^i}{i}\right]$$

$$= \exp\left[\frac{X}{2} + \frac{X^2}{4} - \frac{1}{2}\log(1-X)\right]$$

$$= (1-X)^{-1/2}\exp\left(\frac{X}{2} + \frac{X^2}{4}\right).$$

The function $h(n)$ has several other interesting combinatorial interpretations, e.g., (a) the number of distinct matrices of the form $P + P^{-1}$, where P is an $n \times n$ permutation matrix, and (b) the number of distinct monomials appearing in the expansion of the determinant of an $n \times n$ symmetric matrix whose entries x_{ij} are independent indeterminates (except $x_{ij} = x_{ji}$). For a modification of this result, see [10, Ch. 7.3].

6.6. Example: Suppose we have a room of n children. The children gather into circles by holding hands, and one child stands in the center of each circle. A circle may consist of as little as one child (clasping his or her hands), but each circle must contain a child inside it. In how many ways can this be done? Let this number be $h(n)$. An allowed arrangement of children is obtained by choosing a partition of the children, choosing a child c from each block B to be in the center of a circle, and arranging the other children in the block B in a circle about c. If $|B| = i \geq 2$, then there are $i \cdot (i-2)!$ ways to do this, while there are no ways if $i = 1$. Hence

$$1 + \sum_1^{\infty} h(n)X^n/n! = \exp \sum_2^{\infty}\frac{i \cdot (i-2)!X^i}{i!}$$

$$= \exp X \sum_{i=1}^{\infty}\frac{X^i}{i} = (1-X)^{-X}.$$

The astute reader has undoubtedly realized by now that this example was contrived solely to obtain the curious answer. With a little practice such generating functions can be quickly computed in one's head.

6.7. Example: Let $\mathfrak{S}_n$ denote the group of all permutations of $[n]$. For fixed m $\in$ **P**, let $h(n)$ denote the number of $\rho \in \mathfrak{S}_n$ satisfying $\rho^m = 1$. Such a ρ can be obtained by partitioning $[n]$ into blocks B whose cardinality d divides m, and then choosing a cyclic permutation of B. Such a cyclic permutation can be chosen in $(d-1)!$ ways. Hence

$$1 + \sum_{n=1}^{\infty} h(n)X^n/n! = \exp \sum_{d|m}\frac{(d-1)!X^d}{d!}$$

$$= \exp \sum_{d|m}\frac{X^d}{d}.$$

More generally, the coefficient of $Y_1^{a_1} Y_2^{a_2} \cdots Y_n^{a_n} X^n/n!$ in exp $\sum_{i=1}^{\infty} Y_i X^i/i$ is equal to the number of $\rho \in \mathfrak{S}_n$ with exactly a_i cycles of length i. This well-known result (e.g., [33]) is equivalent to Corollary 6.2 in the case $r = 1$.

6.8. Example: Let $t(n)$ be the number of rooted trees (connected acyclic graphs with a distinguished vertex) on the vertex set $[n]$, and let $f(n)$ be the number of rooted forests (graphs whose components are rooted trees) on $[n]$. The reader who has come this far will instantaneously see that

$$1 + \sum_1^{\infty} f(n)X^n/n! = \exp \sum_1^{\infty} t(n)X^n/n!. \qquad (19)$$

On the other hand, any tree on $[n + 1]$ gives rise to a rooted forest on $[n]$ by removing the vertex $n + 1$ and all incident edges, and putting roots at the vertices adjacent to $n + 1$. Since there are $n + 1$ ways of rooting a tree on $[n + 1]$, we get $(n + 1)f(n) = t(n + 1)$. Setting $T(X) = \sum_{n=1}^{\infty} t(n)X^n/n!$, equation (19) results in the famous functional equation $T(X) = X \cdot \exp T(X)$. Using the Lagrange inversion formula (Theorem 5.6), one easily deduces that $t(n) = n^{n-1}$. For further information on this result, including direct combinatorial proofs, see [29].

6.9. Example: Let $h(n)$ be the number of idempotent functions $\beta:[n] \to [n]$, i.e., $\beta(\beta(i)) = \beta(i)$ for all $i \in [n]$. A function $\beta:[n] \to [n]$ is idempotent if and only if for each $i \in [n]$, the set $\beta^{-1}(i)$ is empty or contains i. Hence we obtain an idempotent function by partitioning $[n]$ and mapping each element of a block B to a fixed element x of that block. If $|B| = i$ then there are i choices for x. Thus

$$\begin{aligned} 1 + \sum_1^{\infty} h(n)X^n/n! &= \exp \sum_1^{\infty} iX^i/i! \\ &= \exp Xe^X. \end{aligned} \qquad (20)$$

For further information on $h(n)$, see [21]. The reader may find it interesting to generalize (20) in various ways. For instance, given $1 \le i < j$, how many functions $\beta:[n] \to [n]$ satisfy $\beta^i = \beta^j$?

6.10. Example: Fix $s > 0$. Let $f_s(n)$ be the number of sequences $\rho_1, \rho_2, \ldots, \rho_s$ of s permutations in the symmetric group $\mathfrak{S}_n$ on $[n]$ which generate a transitive subgroup of $\mathfrak{S}_n$. There are $n!^s$ sequences $\rho_1, \rho_2, \ldots, \rho_s$ with no assumptions on transitivity. Given any such sequence, the orbits of the group generated by $\rho_1, \rho_2, \ldots, \rho_s$ form a partition of $[n]$. Given a partition π of $[n]$ of type $(a_1, a_2, \ldots, a_n)$, the number of sequences $\rho_1, \rho_2, \ldots, \rho_s$ with orbit partition π is clearly $f_s(1)^{a_1} f_s(2)^{a_2} \cdots f_s(n)^{a_n}$. Hence

$$n!^s = \sum_{\pi \in \Pi_n} f_s(1)^{a_1} \cdots f_s(n)^{a_n},$$

so by Corollary 6.2 (with $r = 1$),

$$1 + \sum_1^{\infty} n!^s X^n/n! = \sum_0^{\infty} n!^{s-1} X^n = \exp \sum_1^{\infty} f_s(n)X^n/n!$$

Now let F_s be the free group on generators $x_1, x_2, \ldots, x_s$. Let G be a subgroup of F_s of index n. Let $G_2, G_3, \ldots, G_n$ be an ordering ι (out of the $(n - 1)!$ possible orderings) of the cosets of G not equal to G. Let $G = G_1$. Define permutations $\rho_1, \rho_2, \ldots, \rho_s$ in $\mathfrak{S}_n$ by $x_i G_j = G_{\rho_i(j)}$. It is easily seen that $\rho_1, \rho_2, \ldots, \rho_s$ generate a transitive subgroup of $\mathfrak{S}_n$. It follows from the theory of free groups (e.g., [19, Theorem 7.2.7]) that the map $(\iota, G) \to (\rho_1, \rho_2, \ldots, \rho_s)$ is a bijection between (a) pairs (ι, G), where G is a subgroup of F_s of index n and ι is an ordering of the $n - 1$ proper cosets of G, and (b) sequences $(\rho_1, \ldots, \rho_s) \in \mathfrak{S}_n^s$ whose elements generate a transitive subgroup of $\mathfrak{S}_n$. Thus if $N_s(n)$ denotes the number of subgroups of F_s of index n, then $N_s(n) = f_s(n)/(n - 1)!$ and

$$\sum_{n=1}^{\infty} n!^{s-1} X^n = \exp \sum_{n=1}^{\infty} N_s(n)X^n/n. \qquad (21)$$

A recursion equivalent to (21) appears in [19, Theorem 7.2.9].

From (21) E. Bender [6, §5] has derived an asymptotic expansion for $N_s(n)$ for fixed s.

6.11. Example: Let $n \in \mathbf{P}$ and $s \in \mathbf{N}$, and let $\mathfrak{M}(n, s)$ denote the set of all $n \times n$ matrices of nonnegative integers for which every row and column sums to s. Let $M \in \mathfrak{M}(n, s)$. We regard the rows and columns of M as being indexed by $[n]$; i.e., $M = (m_{ij})$, where $(i, j) \in [n] \times [n]$. By a *k-component* of M, we mean a pair (A, B) of subsets of $[n]$ satisfying the following two properties:

(i) $|A| = |B| = k \geq 1$,

(ii) Let $M(A, B)$ be the $k \times k$ submatrix of M whose rows are indexed by A and whose columns are indexed by B, i.e., $M(A, B) = (m_{ij})$, where $(i, j) \in A \times B$. Then every row and column of $M(A, B)$ sums to s, i.e., $M(A, B) \in \mathfrak{M}(k, s)$.

A component (A, B) is *irreducible* if any component (A', B') with $A' \subset A$ and $B' \subset B$ satisfies $(A', B') = (A, B)$. For instance, $(\{i\}, \{j\})$ is a 1-component (in which case it is irreducible) if and only if $m_{ij} = s$. It is easily seen that the irreducible components of M form a 2-partition of $[n]$. Conversely, any matrix $M \in \mathfrak{M}(n, s)$ can be obtained by choosing a 2-partition π of $[n]$ and "attaching" an irreducible component to each $(A, B) \in \pi$. Let $h_s(a_1, \ldots, a_n)$ denote the number of matrices $M \in \mathfrak{M}(n, s)$ such that M has a_i irreducible i-components (so $n = \Sigma i a_i$). Let $f_s(n)$ be the number of irreducible $n \times n$ matrices $M \in \mathfrak{M}(n, s)$, i.e., $([n], [n])$ is an irreducible component of M. It then follows from Corollary 6.2 that

$$\sum_{n=0}^{\infty} \sum_{a_1, \ldots, a_n} h_s(a_1, \ldots, a_n) Y_1^{a_1} \cdots Y_n^{a_n} X^n / n!^2 = \exp \sum_1^{\infty} f_s(n) Y_n X^n / n!^2. \tag{22}$$

In particular, if $H(n, s) = |\mathfrak{M}(n, s)|$, then

$$\sum_0^{\infty} H(n, s) X^n / n!^2 = \exp \sum_1^{\infty} f_s(n) X^n / n!^2.$$

Similarly, let $H^*(n, s)$ denote the number of matrices in $\mathfrak{M}(n, s)$ with no entry equal to s. Since $f_s(1) = 1$, there follows

$$\sum_0^{\infty} H^*(n, s) X^n / n!^2 = \exp \sum_2^{\infty} f_s(n) X^n / n!^2 = e^{-X} \sum_0^{\infty} H(n, s) X^n / n!^2.$$

It is not difficult to compute $f_2(n)$. Indeed, an irreducible matrix $M \in \mathfrak{M}(n, 2)$ is of the form $P + PQ$, where P is a permutation matrix and Q is a cyclic permutation matrix. There are $n!$ choices for P and $(n-1)!$ choices for Q. If $n > 1$ then P and PQ could have been chosen in reverse order. Hence $f_2(1) = 1$ and $f_2(n) = n!(n-1)!/2$ if $n > 1$. There follows

$$\sum_{n=0}^{\infty} \sum_{a_1, \ldots, a_n} h_2(a_1, \ldots, a_n) Y_1^{a_1} \cdots Y_n^{a_n} X^n / n!^2 = \exp \left[\frac{Y_1 X}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{Y_n X^n}{n} \right].$$

In particular,

$$\sum_0^{\infty} H(n, 2) X^n / n!^2 = (1 - X)^{-1/2} e^{X/2}, \tag{23}$$

$$\sum_0^{\infty} H^*(n, 2) X^n / n!^2 = (1 - X)^{-1/2} e^{-X/2}, \tag{24}$$

Equations (23) and (24) are due to Anand, Dumir, and Gupta [3]. For additional information about the functions $H(n, s)$, see [40], [10, pp. 124–125]. It appears certain, however, that there are no formulas for $H(n, 3)$ as simple as (23). The reader may find it of interest to derive a formula analogous to (23) involving $S(n, 2)$, the number of *symmetric* matrices in $\mathfrak{M}(n, 2)$ (see [18]).

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